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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 20980.43-MA	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) PROGRESS ON H^∞ OPTIMAL SENSITIVITY FOR DELAY SYSTEMS PART I: MINIMUM PHASE PLANT WITH INPUT DELAY. 1 POLE/ZERO WEIGHTING FUNCTION		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER LIDS-P-1513
7. AUTHOR(s) David S. Flamm Sanjoy K. Mitter		8. CONTRACT OR GRANT NUMBER(s) ARO Grant DAAG29-84-K-0005
9. PERFORMING ORGANIZATION NAME AND ADDRESS M.I.T. Laboratory for Information and Decision Systems Cambridge, MA 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE Nov. 22, 1985
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE NA

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Delay Systems
Compensators
Sensitivity Feedback System
Weighting Functions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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November 22, 1985

LIDS-P-1513

Rev. 11/25/85

Progress on $\mathcal{H}^\infty$ Optimal Sensitivity for Delay Systems

Part I: Minimum phase plant with input delay.

1 pole/zero weighting function.

David S. Flamm

Sanjoy K. Mitter

Table of Contents

Summary.

In this report we derive the solution of the $\mathcal{H}^\infty$-minimal weighted sensitivity problem with 1 pole/1 zero sensitivity weighting functions for stable minimum phase rational plants with an input delay. We calculate the explicit feedback compensator and analyze its stability. Although we have a technique for approximating these compensators with proper compensators, the full description of this, along with an analysis of stability robustness will appear in a future report. Another report will also cover generalizations of this work to plants which are unstable, to plants which have right half plane zeros, and to the case of a more general sensitivity weighting function.

Preface.

The results presented here expand upon the work initiated in Appendix B of [Flamm 1985a].

Purpose of Research.

The goal of this work is to obtain and analyze explicit compensators for delay systems which achieve or approximate a closed loop sensitivity function minimal in the $\mathcal{H}^\infty$-norm. A major premise of this investigation is that all real systems contain delays, so that it is

only by examining the optimum for such systems that an understanding of achievable performance can be obtained.

We shall also examine the robustness of the stability of $\mathcal{H}^\infty$-optimal sensitivity compensators with respect to changes in the delay present in the plant. We show below with a simple example that $\mathcal{H}^\infty$-minimal sensitivity feedback systems are not necessarily robustly stable with respect to the addition of a small delay in the feedback loop. This means that there is something wrong with the $\mathcal{H}^\infty$ sensitivity problem as commonly treated. As a consequence of this lack of robustness, the resulting feedback system is ill-posed in the sense of [Willems 1971] pp. 90-91.

We remark here that the ill-posedness for the feedback system seems to be a result of the memoryless part of the opened feedback loop having gain which is too large. See [Willems 1971] p. 100. From this point of view a strictly proper approximation to the optimal compensator for an $\mathcal{H}^\infty$ problem for a rational transfer function plant will likely result in a well-posed system. Nonetheless, it is troublesome to use the formulation of a problem for which the ideal solution is ill-posed.

For plants with a delay in the input, which we shall consider in this report, the optimal feedback systems will be well-posed, according to [Willems 1971] Corollary 4.1.1, precisely because of the delay in the plant. (This conclusion depends upon the nature of the optimal compensators for these systems. We examine this later for the case treated in this report, and see that, although these compensators are unstable, and in fact have infinitely many poles in the right half plane, they are not anticipative.)

We also intend to test the conjecture that designs for systems con

taining delays will provide a means of obtaining compensators which are robust for high frequency phase uncertainty, such as that resulting from right half plane zeros of the plant for which the designer of the compensator is uncertain of the location.

We note that in the example used below to illustrate the ill-posedness of the $\mathcal{H}^{\infty}$ optimal feedback system for the purely rational plant, [Zames and Francis 1983] p. 592 §VI.A, remark that the optimal "Q" parameter is a lead filter. Since this has a certain intuitive appeal (which those authors suggest), we would like to examine how this may extend to the case of delay plants. In particular, we are led to ask, in what manner can we look at the delay in terms of right half plane zeros: for example, as infinitely many zeros at the point ∞?

Finally, we hope to provide conceptual insight into the reason delays limit the sensitivity of a compensated system.

We also regard the delay problems considered here as a first step towards considering similar design issues for other infinite dimensional plants.

An Example of Ill-posedness of the $\mathcal{H}^{\infty}$-minimal Sensitivity Feedback System for a Rational Plant.

The importance of understanding $\mathcal{H}^{\infty}$-minimal sensitivity design for delay systems is emphasized by the fact that designs which achieve $\mathcal{H}^{\infty}$-minimal sensitivity for finite dimensional systems are not generally stable when a small delay is added.[1] We illustrate this fact with a

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[1]This point was suggested by Gunter Stein.

simple example.

In [Zames and Francis 1983, p. 593] the solution to the $\mathcal{H}^\infty$-minimal sensitivity problem is computed for the case of a stable plant with two right half plane zeros. We consider this example here. Let

$$P_0 = \frac{(b_1-s)(b_2-s)}{(b_1+s)(b_2+s)} \cdot P_1, \text{ with } \mathfrak{Re}(b_i) > 0 \text{ and } b_1+b_2 \in \mathbb{R},$$

be the plant, where P_1 is stable and minimum phase, and let

$$W = \frac{s+1}{s+\beta}, \text{ with } \beta > 0,$$

be the sensitivity weighting function. The optimal "Q" parameter is

$$\widetilde{Q}(s) = \left[1 - \frac{D(c-s)}{W(s)(c+s)}\right] \cdot \frac{(b_1+s)(b_2+s)}{(b_1-s)(b_2-s)} \cdot P_1^{-1},$$

where D and c are constants determined by β and the b_is. <u>D may be positive or negative, depending on whether $\beta > 1$ or $\beta < 1$.</u> (We will not show it here, but $\beta < 1$ is the only interesting case.) We assume D is positive, for $\beta < 1$. Then the optimal feedback compensator is
$F = \widetilde{Q}(1-P\widetilde{Q})^{-1}$.

Now suppose that the true plant is really $P(s) = e^{-\epsilon s}P_0(s)$, with $\epsilon > 0$. We check stability for the closed loop system with true plant and compensator designed for the nominal plant P by computing $\frac{P}{1+PF}$ and checking the location of its poles. A straightforward computation gives

$$\frac{P}{1+PF} = P \cdot \frac{e^{s\epsilon}D(c-s)}{e^{s\epsilon}D(c-s) + [W(s)(c+s)-D(c-s)]}.$$

We now show that this is unstable by showing that the denominator, call it M(s), in the above expression has infinitely many right half plane zeros.

For this we evaluate M(s) on the imaginary axis and separate it into real and imaginary parts, $M(i\omega) = F(\omega) + i \cdot G(\omega)$. Now we show that $F(\omega)$ has only finitely many real zeros, and conclude by appealing to [Pontryagin 1955], Theorems 3 and 6, that D(s) has zeros in the right

half plane.

A computation gives

$$F(\omega) = \omega^2 D[\cos(\epsilon\omega) - \frac{D+1}{D}] + \omega D(\beta-c)\sin(\epsilon\omega) + \beta cD[\cos(\epsilon\omega)-1] + c.$$

Since $D > 0$, this equation cannot be satisfied for sufficiently large ω, and we conclude that $F(\omega)$ has only finitely many real zeros. Therefore, $D(s)$ has zeros in the right half plane.

Problem Considered.

We consider the $\mathcal{H}^\infty$ optimal sensitivity control problem formulated in [Zames 1981], but with plants of the form $P(s) = A(s)+e^{-s\Delta}B(s)$, where A and B are proper rational functions, and $\Delta > 0$. The block diagram in Figure 1 shows the feedback system we are considering.
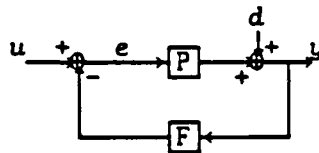
Figure 1. Feedback system considered.

The problem is to minimize the $\mathcal{H}^\infty$ norm of the weighted closed loop sensitivity, $X(s) = W(s)[1+P(s)F(s)]^{-1}$ over all stabilizing feedbacks $F(s)$.

In this work we consider only weighting functions which are rational and bounded away from zero at ∞.[2] The reason for the rationality assumption is simple convenience, and the fact that nothing seems to be

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[2] The importance of this was suggested by Gunter Stein.

lost from this restriction. If, however, strictly proper weighting functions are considered, since the resulting optimal weighted sensitivities will be all-pass, the corresponding unweighted sensitivities will be unbounded at ∞. This would be an unacceptable design, were it really to be used. As above for the question of ill-posedness, it could be argued that real implementations will use proper approximations to the optimal compensators. Then the unweighted sensitivity will not only be bounded at infinity, it will have value 1 there. However, since the approximation techniques presented in the literature, [Zames and Francis 1983] and [Vidyasagar 1985], achieve sensitivity with norm arbitrarily close to the optimal by approximating the optimal weighted sensitivity to higher and higher frequencies, these techniques cause the unweighted sensitivity to grow as the approximation improves. A trade-off would then be required. Since bounded sensitivity is a real design constraint, which proper choice of non-strictly proper weighting function accommodates, the use of such a weighting function seems to be the natural way of providing the trade-off.

Our solution will start with a more narrow problem, and proceed stepwise to generalize the solution. The sequence of problems to be considered is:

A. A=0 and B is minimum phase and stable, with sensitivity weighting function W having one pole and one zero. (A=0 is the case of an *input delay*.)

B. W is any stable rational minimum phase weighting function.

C. B can have right half plane zeros.

D. B can be unstable.

E. A is non-zero.

Since the optimal compensator will in general be non-proper, we also consider the problem

F. Approximation of the optimum compensator with proper ones.

As part of our solution, we will examine

G. The stability properties of the feedback compensators we obtain, and

H. The robustness of stability of the resulting closed loop system with respect to perturbation of the delay interval and pole/zero locations of the plant.

In this report we discuss problem A, and the issues in F and G as applied to A. Subsequent reports will cover the other problems.

Derivation of Optimal Sensitivity.

Formulation. The calculation of the optimum sensitivity for this case was first presented in Appendix B of [Flamm 1985a]. The following re-derivation is presented in a form that lends itself to generalization to the succeeding problems.

The transformation of the problem to the form

$$\inf_{H \in \mathscr{H}_\infty} \|W(s) - \psi H(s)\|_\infty \tag{1}$$

is standard. See, for example, [Zames and Francis 1983]. Here $W(s)$ is the sensitivity weighting function and ψ is the inner factor of the plant. (We have assumed that the plant is stable, and that the weighting function is stable and minimum phase.) In problem A, $W(s) = \frac{s+1}{s+\beta}$ and $\psi = e^{-s\Delta}$. Note that we have normalized the weighting function so that

its zero is at the point -1.

Our approach is based on [Sarason 1967], to which the reader is referred for the mathematical justification of our work. Since we shall deal with operators which are not compact, we cite Theorem 1 on page 179 in that paper for the general case. However, since we subsequently establish that the particular operators in which we are most interested have maximal vectors, it is really Proposition 5.1 on p. 188 that we require. The work in this report is further substantially guided by the whole of §7, pp. 191-194.

In order to follow [Sarason 1967], we view $\left[W(s)-\psi(s)H(s)\right]$ as an operator on $\mathcal{H}^2$. The compression of this operator to $K = \mathcal{H}^2 \ominus \psi\mathcal{H}^2$ is equal to the compression of $W(s)$ on the same subspace. Call this latter operator $T = \pi_K W|_K$. The infimum in (1) cannot be less than the operator norm of T. Theorem 1 in [Sarason 1967] says that the desired infimum is in fact equal to $\|T\|$.

Solution Technique. In order to calculate this norm, we first characterize $K = (\psi\mathcal{H}^2)^{\perp}$. Suppose f, $h \in L^2(0,\infty)$ and $\hat{\cdot}$ denotes Laplace transform, $\hat{g} = \mathcal{L}(g)$. If $\hat{f}(s)\in K$, then

$$0 = \langle\hat{f}(s), e^{-s\Delta}\hat{h}(s)\rangle_{\mathcal{H}^2}, \text{ for all } \hat{h}\in\mathcal{H}^2$$
$$= \langle\hat{f}(i\omega), e^{-i\omega\Delta}\hat{h}(i\omega)\rangle_{L^2}$$
$$= \langle f(t), h(t-\Delta)\rangle_{L^2}$$

by Parseval's theorem. Therefore $f(t) = 0$ a.e. $[\Delta,\infty)$. Conversely, if $f(t)\in L^2$ and $supp(f)\subseteq[0,\Delta]$, then $\hat{f}\in(\psi\mathcal{H}^2)^{\perp}$. Thus

$$K = (\psi\mathcal{H}^2)^{\perp} = \langle\hat{f}:f\in L^2[0,\infty) \text{ and } supp(f)\subseteq[0,\Delta]\rangle$$
$$= \left[\frac{1-e^{-s\Delta}}{s}\right]\times\mathcal{H}^2$$

where "$*$" denotes convolution. Accordingly, for $\hat{f} \in K$,

$$T\hat{f} = \mathcal{L}\left[[u(t)-u(t-\Delta)]\int_0^t w(t-\tau)f(\tau)d\tau\right], \text{ where } w = \mathcal{L}^{-1}(W).$$

It is easy to see $\pi_K(W|_K) = \frac{1-e^{-s\Delta}}{s}*W(s)|_K$. So

$$\|T_W\| = \sup_{\substack{h \in \mathcal{H}^2 \\ \left\|\frac{1-e^{-s\Delta}}{s}*h\right\|_2=1}} \|\frac{1-e^{-s\Delta}}{s}*W(s)(\frac{1-e^{-s\Delta}}{s}*h)\|_2$$

$$= \sup \|[w(t)*(u(t)-u(t-\Delta))h(t)](u(t)-u(t-\Delta))\|_2$$

$$= \sup_{\substack{supp(\ell)\subseteq[0,\Delta] \\ \|\ell\|_2=1}} \|(w(t)*\ell(t))(u(t)-u(t-\Delta))\|_2$$

$$= \sup \left(\int_0^\Delta |w(t)*\ell(t)|^2 dt\right)^{1/2} = \|V\|$$

where V is the operator on $L^2(0,\Delta)$ defined by

$$(Vf)(t) = \int_0^t w(t-\tau)f(\tau)d\tau. \tag{2}$$

A way to find this supremum is to use the facts that $\|V\|^2 = \|V^*V\|$ (via the definition of adjoint) and that $\rho(V^*V) = \|V^*V\|$ since V^*V is normal [Rudin 1973, p. 282]. Therefore $\|V\| = \rho(V^*V)^{1/2}$. If V is compact we need only find the largest eigenvalue of V^*V, but this will not be the general case. However, since in our case V is the identity plus a compact operator, we have only slightly more complication: Since we shall have V^*V-I is compact, we know from Weyl's theorem that the spectrum of V^*V and I differ only by eigenvalues. [Halmos 1967, pp. 92 &

295]. Therefore, $\sigma(V^*V) \subseteq \sigma(V^*V-I) \cup \{1\}$.

Thus we can find the norm of T by finding the largest eigenvalue of V^*V. We do this is two steps:

- Compute V^*V.

- Solve the eigenvalue/function problem for this operator by transforming the eigenvalue-defining equation to a differential equation.

We note that the operator V is equivalent to T via the Laplace transformation. In particular, there is a one-to-one correspondence between eigenvectors, and the eigenvalues are the same. Furthermore, compactness of V is equivalent to compactness of T.

Computation of V^*V. V^* is defined by $\langle x, V^*y \rangle = \langle Vx, y \rangle$, so we just compute:

$$\langle Vx, y \rangle = \int_0^\Delta y(t) \cdot (w(t)*x(t))dt \qquad \begin{array}{l}\text{[We have used the fact that} \\ \text{both } w \text{ and } y \text{ have their} \\ \text{support on } [0,\infty).]\end{array}$$

Using the fact that

$$w(t-\tau) = \delta_{(t-\tau)} + w_0(t-\tau) = \delta_{(t-\tau)} + (1-\beta)\cdot e^{-\beta(t-\tau)}$$

we then have

$$\int_0^\Delta y(t)\cdot(w(t)*x(t))dt = \int_0^\Delta y(t)\cdot[\int_0^t w_0(t-\tau)x(\tau)d\tau + \delta_t*x(t)]dt$$

$$= \int_0^\Delta y(t)\cdot[(1-\beta)\cdot e^{-\beta t}\cdot\int_0^t e^{\beta\tau}\cdot x(\tau)d\tau + x(t)] \, dt$$

$$= \left[\left(\int_0^t y(\tau) e^{-\beta \tau} d\tau \right) \left(\int_0^t e^{\beta \tau} \cdot x(\tau) d\tau \right) \right]_0^\Delta -$$

$$\int_0^\Delta \left[\int_0^t y(\tau) e^{-\beta \tau} d\tau \right] \cdot [e^{\beta t} \cdot x(t)] \cdot dt + \int_0^\Delta y(t) x(t) dt$$

$$= \int_0^\Delta x(t) \int_0^\Delta y(\tau) w_0(\tau - t) d\tau dt - \int_0^\Delta x(t) \int_0^t y(\tau) w_0(\tau - t) d\tau dt$$

$$+ \int_0^\Delta x(t) [\delta(t) * y(t)] dt$$

$$= \int_0^\Delta x(t) \left[\int_t^\Delta y(\tau) w_0(\tau - t) d\tau + (\delta * y)(t) \right] dt$$

$$= \langle \int_t^\Delta y(\tau) w(\tau - t) d\tau, x(t) \rangle$$

and we conclude

$$V^* y = \int_t^\Delta y(\tau) w(\tau - t) d\tau. \tag{3}$$

Solution of eigenvalue problem. We want to solve

$$\lambda^2 f = V^* V f = (V_0^* + \delta) * [(V_0 + \delta) * f]$$
$$= (V_0^* V_0 + V_0^* + V_0 + I)(f). \tag{4}$$

Now let

$$y = V f = \int_0^t w(t - \tau) f(\tau) d\tau \text{ for } t \in [0, \Delta],$$

and let

$$z = V^* y = \int_t^\Delta w(\tau - t) y(\tau) d\tau \text{ for } t \in [0, \Delta].$$

Since $w(t) = \delta_t + (1-\beta) \cdot e^{-\beta t}$, we can take

$$\frac{d}{dt} x_1 = -\beta \cdot x_1 + f$$
$$y = (1-\beta) \cdot x_1 + f$$
$$\frac{d}{dt} x_2 = \beta \cdot x_2 - (1-\beta) \cdot y$$
$$z = x_2 + y$$

as a state space model for V^*V valid on $[0, \Delta]$.

Boundary conditions are given by $y(0) = f(0)$ and $z(\Delta) = y(\Delta)$. This is equivalent to $x_1(0) = 0$ and $x_2(\Delta) = 0$ (if $\beta \neq 1$: we see later that this is no restriction).

More concisely,

$$\dot{x} = \begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ \beta-1 \end{bmatrix} \cdot f \tag{5a}$$

$$z = [1-\beta \quad 1] \cdot x + f \tag{5b}$$

with $\qquad 0 = \begin{bmatrix} x_1(0) \\ x_2(\Delta) \end{bmatrix}.$ $\tag{5c}$

Now we set $z = \lambda^2 f$ in order to find the eigenfunctions and eigenvalues of V^*V. Then

$$(\lambda^2 - 1)f = [1-\beta \quad 1] \cdot x \tag{6}$$

replaces (5b).

Differentiating this we get

$$(\lambda^2 - 1)\dot{f} = [1-\beta \quad 1] \cdot \dot{x}$$
$$= [1-\beta \quad 1] \cdot \left[\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ \beta-1 \end{bmatrix} \cdot f \right]$$
$$= [1-\beta \quad 1] \cdot \begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix} \cdot x \tag{7}$$

Differentiating again, we get

$$(\lambda^2-1)\ddot{f} = [1-\beta \quad 1]\cdot\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix}\cdot\left(\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix}\cdot x + \begin{bmatrix} 1 \\ \beta-1 \end{bmatrix}\cdot f\right)$$

$$= [1-\beta \quad 1]\cdot\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix}^2\cdot x + [1-\beta \quad 1]\cdot\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix}\cdot\begin{bmatrix} 1 \\ \beta-1 \end{bmatrix}\cdot f$$

$$= [1-\beta \quad 1]\cdot\begin{bmatrix} \beta^2 & 0 \\ 0 & \beta^2 \end{bmatrix}\cdot x + [1-\beta \quad 1]\cdot\begin{bmatrix} -\beta & 0 \\ -(1-\beta)^2 & \beta \end{bmatrix}\cdot\begin{bmatrix} 1 \\ \beta-1 \end{bmatrix}\cdot f$$

$$= \beta^2\cdot(z-f) + (\beta^2-1)f$$

$$= \beta^2\cdot(\lambda^2-1)\cdot f - f + \beta^2 f$$

$$= (\beta^2\lambda^2 - 1)\cdot f.$$

If $\lambda^2 = 1$ then $(\beta^2-1)f = 0$ implies $f = 0$ or $\beta^2 = 1$. This latter case implies $\beta = \pm 1$, and then our frequency domain weighting function must be $W(s) = \frac{s-1}{s+1}$ or $W(s) = 1$. These are of no interest since we only allow weighting functions which are non-constant and outer. Thus we can assume that $\lambda^2 \neq 1$, and we get

$$\ddot{f} = \frac{\beta^2\lambda^2-1}{\lambda^2-1}\cdot f. \tag{8}$$

Boundary conditions follow from (5c), (6) and (7):

$$(\lambda^2-1)\cdot f(\Delta) = [1-\beta \quad 1]\cdot x(\Delta)$$
$$= (1-\beta)x_1(\Delta)$$
$$= (1-\beta)\cdot\int_0^\Delta e^{-\beta(\Delta-\tau)}f(\tau)d\tau$$
$$(\lambda^2-1)\dot{f}(0) = [-2(1-\beta)^2 \quad \beta]\cdot x(0)$$
$$= \beta\cdot x_2(0)$$
$$= \beta\cdot(\lambda^2-1)\cdot f(0).$$

Thus
$$f(\Delta) = \frac{1-\beta}{\lambda^2-1}\int_0^\Delta e^{-\beta(\Delta-\tau)}f(\tau)d\tau \tag{9a}$$

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and
$$\dot{f}(0) = \beta f(0). \qquad\qquad (9b)$$

Now it appears that we could have

$$(i)\ \frac{\beta^2\lambda^2-1}{\lambda^2-1} = 0,\quad (ii)\ \frac{\beta^2\lambda^2-1}{\lambda^2-1} > 0\ \text{or}\ (iii)\ \frac{\beta^2\lambda^2-1}{\lambda^2-1} < 0.$$

We consider each of these cases in turn.

Case (i). This implies $\ddot{f}=0$. Then $f(t)=k\cdot t + m$, but $\dot{f}(0)=\beta f(0)$

implies $f(t)=m(\beta t+1)$, and we normalize by assuming $m=1$. Then (i) gives

$$\lambda^2\beta^2 = 1,$$

or

$$\lambda^2 = \frac{1}{\beta^2}.$$

The boundary condition at $t = \Delta$ gives us

$$\beta\cdot\Delta + 1 = \frac{1-\beta}{\lambda^2-1}\int_0^\Delta e^{-\beta(\Delta-\tau)}(\beta\tau+1)d\tau = \frac{(1-\beta)e^{-\beta\Delta}}{\lambda^2-1}\int_0^\Delta (\beta\tau+1)e^{\beta\tau}d\tau.$$

This implies

$$\frac{(\beta\Delta+1)(\lambda^2-1)}{(1-\beta)e^{-\beta\Delta}} = \int_0^\Delta (\beta\tau+1)e^{\beta\tau}d\tau = \left[\tau e^{\beta\tau}\right]_0^\Delta - \int_0^\Delta e^{\beta\tau}d\tau + \int_0^\Delta e^{\beta\tau}d\tau$$

$$= \Delta e^{\beta\Delta}$$

Thus $(\lambda^2-1) = \Delta\cdot\frac{1-\beta}{\beta\Delta+1}$, and substituting $\lambda^2 = \frac{1}{\beta^2}$ we get

$(\Delta+1)\cdot\beta^2 - \Delta\beta - 1 = 0$. This implies that

$$\beta = \frac{\Delta\pm\sqrt{\Delta^2+4\cdot(\Delta+1)}}{2\cdot(\Delta+1)} = \langle 1, \frac{-1}{\Delta+1}\rangle.$$

By assumption $\beta>0$, and so we must have $\beta=1$. But then $W(s) = \frac{s+1}{s+1} = 1$.

Thus case (i) is excluded.

<u>Case (ii)</u>. In this case

$$f = e^{kt} + a \cdot e^{-kt} \text{ with } k^2 = \left[\frac{\beta^2 \lambda^2 - 1}{\lambda^2 - 1}\right].$$

For later reference we note this implies

$$\lambda^2 = \frac{k^2 - 1}{k^2 - \beta^2}. \tag{10}$$

We assume without loss of generality that $k > 0$. Then the boundary condition (9b) gives us

$$k - a \cdot k = \beta \cdot (1 + a), \text{ or } a = \frac{k - \beta}{k + \beta}. \tag{11}$$

Boundary condition (9a) gives

$$
\begin{aligned}
e^{\Delta k} - a \cdot e^{-\Delta k} &= \frac{1-\beta}{\lambda^2 - 1} \int_0^\Delta e^{-\beta(\Delta - \tau)} \cdot (e^{k\tau} + a \cdot e^{-k\tau}) d\tau \\
&= \frac{1-\beta}{\lambda^2 - 1} e^{-\beta\Delta} \cdot \int_0^\Delta \left[e^{(k+\beta)\tau} + a \cdot e^{(\beta - k)\tau} \right] d\tau \\
&= \frac{1-\beta}{\lambda^2 - 1} e^{-\beta\Delta} \cdot \left[\frac{e^{(k+\beta)\Delta}}{k+\beta} + \frac{a \cdot e^{(\beta - k)\Delta}}{\beta - k} - \frac{1}{k+\beta} - \frac{a}{\beta - k} \right] \\
&= \frac{1-\beta}{\lambda^2 - 1} e^{-\beta\Delta} \cdot \left[\frac{e^{(k+\beta)\Delta}}{k+\beta} - \frac{e^{(\beta - k)\Delta}}{k+\beta} \right].
\end{aligned}
$$

using (11). A little algebra gives us

$$e^{\Delta k} - \frac{k-\beta}{k+\beta} e^{-\Delta k} = \frac{1-\beta}{\lambda^2 - 1} \frac{1}{k+\beta} \left[e^{k\Delta} - e^{-k\Delta} \right]$$

and

$$k \cdot (e^{k\Delta} - e^{-k\Delta}) + \beta \cdot (e^{k\Delta} + e^{-k\Delta}) = \frac{1-\beta}{\lambda^2 - 1} (e^{k\Delta} - e^{-k\Delta})$$

so

$$k + \beta \cdot \coth(k \cdot \Delta) = \frac{1-\beta}{\lambda^2-1}.$$

Substituting for λ^2 from (10) we get

$$0 = k + \beta \cdot \coth(k \cdot \Delta) + \frac{k^2-\beta^2}{\beta+1}$$

$$= k + \beta \cdot \coth(k \cdot \Delta) + (k+\beta) \cdot \frac{k-\beta}{\beta+1}$$

$$= k + \beta \cdot \coth(k \cdot \Delta) + (k+\beta) \cdot (\frac{k+1}{\beta+1} - 1)$$

$$> 0$$

since $k>0$, $\Delta>0$, and therefore $\coth(k\Delta) > 1$. This is a contradiction, and thus case (ii) does not occur.

Case (iii). In this case

$$f(t) = \cos(\omega t + \varphi) \text{ with } \omega^2 = -(\frac{\beta^2\lambda^2-1}{\lambda^2-1}). \tag{12}$$

Note that this definition of ω^2 implies $\lambda^2 = 1 + \frac{1-\beta^2}{\omega^2+\beta^2}$. Evaluating the initial condition in (9b) we see $\dot{f}(0) = \beta f(0)$ implies $-\omega\sin(\varphi) = \beta\cos(\varphi)$. $\omega \neq 0$ implies $\cos(\varphi) \neq 0$. So $\tan(\varphi) = \frac{\beta}{\omega}$, and therefore

$$\boxed{\frac{\beta}{(\beta^2+\omega^2)^{\frac{1}{2}}} = -\sin(\varphi) \text{ and } \frac{\omega}{(\beta^2+\omega^2)^{\frac{1}{2}}} = \cos(\varphi)} . \tag{13}$$

The initial condition in (9a) gives

$$\cos(\omega\Delta + \varphi) = f(\Delta) = \frac{1-\beta}{\lambda^2-1} e^{-\beta\Delta} \int_0^\Delta e^{\beta\tau}\cos(\omega\tau + \varphi)d\tau$$

$$= \frac{1-\beta}{\lambda^2-1} e^{-\beta\Delta} \cdot [\frac{\omega\sin(\Delta\omega+\varphi) + \beta\cos(\Delta\omega+\varphi)}{e^{-\beta\Delta}(\omega^2+\beta^2)} - \frac{\omega\sin(\varphi) + \beta\cos(\varphi)}{\omega^2+\beta^2}]$$

$$= \frac{1-\beta}{\lambda^2-1} [\frac{\omega\sin(\omega\Delta+\varphi) + \beta\cos(\Delta\omega+\varphi)}{\omega^2+\beta^2} - e^{-\beta\Delta} \cdot \frac{\omega\sin(\varphi) + \beta\cos(\varphi)}{\omega^2+\beta^2}]$$

$$= \frac{1-\beta}{\lambda^2-1} \cdot \frac{1}{(\beta^2+\omega^2)^{\frac{1}{2}}} \Big[\cos(\varphi)\sin(\omega\Delta+\varphi) - \sin(\varphi)\cos(\omega\Delta+\varphi)$$

$$- e^{-\beta\Delta}(\cos(\varphi)\sin(\varphi) - \sin(\varphi)\cos(\varphi)) \Big]$$

$$= \frac{1-\beta}{\lambda^2-1} \cdot \frac{\sin(\omega\Delta)}{(\beta^2+\omega^2)^{\frac{1}{2}}}$$

Expanding the left hand side, we get

$$\cos(\omega\Delta)\cos(\varphi) - \sin(\Delta\omega)\sin(\varphi) = \frac{(1-\beta)\sin(\omega\Delta)}{(\lambda^2-1)\cdot(\omega^2+\beta^2)^{\frac{1}{2}}}$$

Combining this with the conditions from (13) we get

$$\omega\cdot\cos(\omega\Delta) + \beta\cdot\sin(\omega\Delta) = \frac{(1-\beta)\sin(\omega\Delta)}{\lambda^2-1}. \tag{14}$$

If $\cos(\omega\Delta)=0$, (14) implies that $\beta = \frac{1-\beta}{\lambda^2-1}$, and therefore $\lambda^2 = \frac{1}{\beta}$. But we must also have from (12) $\lambda^2 = 1 + \frac{1-\beta^2}{\omega^2+\beta^2}$, and so $\frac{1}{\beta} = 1 + \frac{1-\beta^2}{\omega^2+\beta^2}$, which implies $\omega^2 = \beta$. We conclude that $\cos(\omega)=0$ only if $\omega^2 = \beta$. Conversely, if $\omega^2 = \beta$, (12) implies $\lambda^2 = \frac{1}{\beta}$, and so (14) implies $\omega\cdot\cos(\omega\Delta) = 0$. Then $\cos(\omega\Delta) = 0$ since we have assumed $\omega \neq 0$. So $\cos(\omega\Delta) = 0$ is equivalent to $\omega^2 = \beta$. But $\cos(\omega\Delta) = 0$ amounts to $\omega\Delta = \frac{2n+1}{2}\pi$, an then from (13) $\tan(\varphi) = -\sqrt{\beta}$. So $\lambda^2 = \frac{1}{\beta} = \frac{1}{\omega^2} = \left[\frac{2\Delta}{(2n+1)\pi}\right]^2$ is an eigenvalue when $\beta = \left[\frac{(2n+1)\pi}{2\Delta}\right]^2$. We see below that when $\beta = \left[\frac{\pi}{2\Delta}\right]^2$ and we also have $\frac{2\Delta}{\pi} > 1$, it is the largest eigenvalue.

If $\cos(\omega\Delta) \neq 0$, then we have

$$\omega = -(\beta\frac{1-\beta}{\lambda^2-1})\tan(\omega\Delta) = \frac{\lambda^2\beta-1}{\lambda^2-1}\tan(\omega\Delta).$$

Substituting from (12) for (λ^2-1) we get

$$\omega = \frac{\omega^2 - \beta}{1+\beta} \tan(\omega\Delta).$$

Since $\cos(\omega\Delta) \neq 0$ implies $\omega^2 \neq \beta$, we conclude

$$\boxed{\tan(\omega\Delta) = \frac{(1+\beta)\omega}{\omega^2 - \beta}} \,. \tag{15}$$

For given β we can numerically solve this equation for ω, finding multiple solutions. From the definition of ω^2 in (12), these solutions give us the eigenvalues of V^*V via $\lambda^2 = 1 + \frac{1-\beta^2}{\omega^2 + \beta^2} = \frac{\omega^2 + 1}{\omega^2 + \beta^2}$. We wish to pick the one among solutions to the equation for ω that gives the largest eigenvalue. We see that there are two cases:

- If $\beta < 1$ then we should pick the smallest solution for ω ($\omega=0$ has been excluded by our consideration of case (iii)).
- If $\beta > 1$, we should pick the largest solution for ω, but one can see from Fig. 1 that there is no upper bound on solutions to (15). Thus there is an infinite sequence of eigenvalues approaching $\lambda^2 = 1$ from below. (This means that V^*V is not compact, as we already knew.)

As pointed out above, the spectrum of V^*V is the set of eigenvalues augmented possibly by $\{1\}$. Thus for $\beta > 1$, the spectral radius of V^*V is 1, and therefore $\|V\| = 1$.

For $\beta < 1$, we also have $1 \in \sigma(V^*V)$, but this does not affect the spectral radius since the largest eigenvalue is greater than 1. It is easy

to see that the eigenvalues of V^*V are contained in $[\lambda^2_{min}, 1)$ for $\beta > 1$, and in $(1, \lambda^2_{max}]$ for $\beta < 1$.

Since the open loop system has sensitivity of norm 1 for the case $\beta > 1$, the optimal sensitivity is attained by $H=0$ in (1), that is, zero feedback. In other words, $\|W\|_\infty = 1$ when $\beta > 1$, and this is the infimal sensitivity according to the above argument. (We note that W is not inner. For non-compact operators, minimal dilations are not necessarily unique or inner.)

To summarize the situation, we have found two cases of interest: either there is an eigenvalue of V^*V equal to the norm of T, or $\|T\|=1$, and we can find a sequence of eigenvalues of V^*V approaching 1. This will be used below to compute the minimal dilation of T to $\mathfrak{X}^2$ in the first case. In the second case W itself is a minimal dilation of T to $\mathfrak{X}^2$.

For later reference we note that, using the definition of λ^2 from (12), (14) implies

$$\beta = \omega \cdot \frac{\omega \cdot \sin(\omega\Delta) - \cos(\omega\Delta)}{\sin(\omega\Delta) + \omega \cdot \cos(\omega\Delta)} . \tag{16}$$

The solutions of the equation $\tan(\omega\Delta) = \dfrac{(1+\beta)\omega}{\omega^2 - \beta}$ can be characterized graphically as indicated in Figure 2. (Notice that we always have $\beta > 0$.)
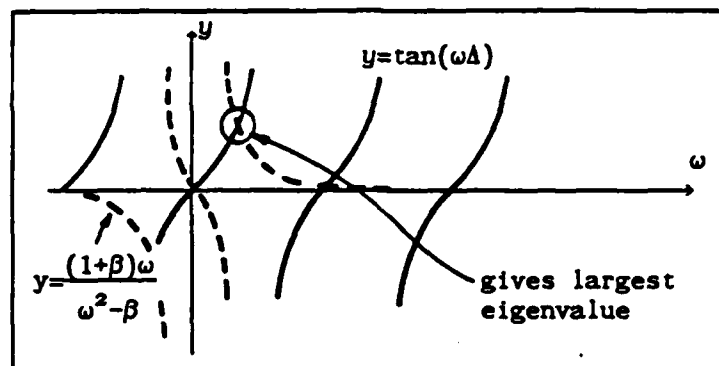
Figure 2. Graphical solution of (15).

From this drawing we can see that the case $\beta = \left[\frac{(2n+1)\pi}{2\Delta}\right]^2$ corresponds to the coincidence of vertical asymptotes of $y = \tan(\Delta\omega)$ and $y = \frac{(1+\beta)\omega}{\omega^2-\beta}$. If $\beta = \left[\frac{\pi}{2\Delta}\right]^2$ the asymptote will be the one closest to the origin for $\tan(\Delta\omega)$. Then the other eigenvalues will correspond to larger values of ω, and thus will be smaller when $\beta < 1$.

Calculation of optimal sensitivity for $\beta < 1$. As part of this solution for $\beta < 1$ we obtain the eigenfunction f for the largest eigenvalue of V^*V. This f is a maximal vector for V, and so we can follow the method in Sarason to compute the minimal dilation of T to $\mathcal{H}^2$.

The following applies to all eigenfunctions f, but gives us the minimal dilation of T only for the case where f is a maximal vector for V.

Let ω_0 be a solution to (15). When ω_0 corresponds to a maximal vector, according to the proof of Proposition 5.1 in [Sarason 1967], $T/\|T\|_\infty$ will be interpolated by an inner function given by $\frac{T\hat{f}}{\|T\|_\infty\hat{f}}$, where $\hat{}$:

denotes Laplace transform.   In any case, the formula below for $\frac{T\hat{f}}{\hat{f}}$ will be valid.   First we calculate

$$Vf = \int_0^t [\delta_{(t-\tau)} + (1-\beta)e^{-\beta(t-\tau)}]\cos(\omega_0\tau+\varphi)d\tau$$

$$= \cos(\omega_0 t+\varphi) + (1-\beta)\int_0^t e^{-\beta(t-\tau)}\cos(\omega_0\tau+\varphi)d\tau$$

$$= \cos(\omega_0 t+\varphi) + (1-\beta)\frac{\sin(\omega_0 t)}{(\omega_0^2+\beta^2)^{\frac{1}{2}}}$$

Then

$$\frac{T\hat{f}}{\hat{f}} = \frac{\mathcal{L}(Vf)}{\mathcal{L}(f)}$$

$$= 1 + \frac{1-\beta}{(\omega_0^2+\beta^2)^{\frac{1}{2}}} \cdot \frac{\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))]}{\mathcal{L}[\cos(\omega_0 t+\varphi)\cdot(u(t)-u(t-\Delta))]}$$

$$= 1 + \frac{(1-\beta)\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))]}{\mathcal{L}[(\omega_0\cos(\omega_0 t)+\beta\sin(\omega_0 t))\cdot(u(t)-u(t-\Delta))]} \tag{17}$$

$$\mathcal{L}[\sin(\omega_0 t)\cdot(u(t)-u(t-\Delta))] = \frac{1-e^{-s\Delta}}{s} * \frac{\delta_{(s-i\omega)} - \delta_{(s+i\omega)}}{2i}$$

$$= \frac{1}{2i}[\frac{1-e^{-(s-i\omega_0)\Delta}}{s-i\omega_0} - \frac{1-e^{-(s+i\omega_0)\Delta}}{s+i\omega_0}]$$

$$= \frac{2i\omega_0 - e^{-s\Delta}[-(s+i\omega_0)e^{i\omega_0\Delta}+(s-i\omega_0)e^{-i\omega_0\Delta}]}{2i(s^2+\omega_0^2)}$$

$$= \frac{\omega_0 + e^{-s\Delta}[s\cdot\sin(\omega_0\Delta)+\omega_0\cdot\cos(\omega_0\Delta)]}{\omega^2-\omega_0^2} \tag{18}$$

$$\mathcal{L}[\cos(\omega_0 t)\cdot(u(t)-u(t-\Delta))] = \frac{1-e^{-s\Delta}}{s} * \frac{\delta_{(s-i\omega_0)} + \delta_{(\omega+i\omega_0)}}{2}$$

$$= \frac{1}{2}[\frac{1-e^{-(s-i\omega_0)\Delta}}{s-i\omega_0} + \frac{1-e^{-(s+i\omega_0)\Delta}}{s+i\omega_0}]$$

$$= \frac{s - e^{-s\Delta}[s\cdot\cos(\omega_0\Delta) - \omega_0\cdot\sin(\omega_0\Delta)]}{s^2+\omega_0^2} \tag{19}$$

Using (18) and (19) in (17), we get

$$\frac{T\hat{f}}{\hat{f}} = 1 +$$

$$\frac{(1-\beta)[-\omega_0 + e^{-s\Delta} \cdot (s \cdot \sin\omega_0\Delta + \omega_0\cos\omega_0\Delta)]}{\omega_0[-s + e^{-s\Delta}(s \cdot \cos\omega_0\Delta - \omega_0\sin\omega_0\Delta)] + \beta[-\omega_0 + e^{-s\Delta}(s \cdot \sin\omega_0\Delta + \omega_0\cos\omega_0\Delta)]}$$

$$= 1 + \frac{(1-\beta)[-\omega_0 + e^{-s\Delta}(s \cdot \sin\omega_0\Delta + \omega_0\cos\omega_0\Delta)]}{-\omega_0(\beta+s) + e^{-s\Delta}[s(\omega_0\cos\omega_0\Delta + \beta\sin\omega_0\Delta) + \omega_0(\beta\cos\omega_0\Delta - \omega_0\sin\omega_0\Delta)]}. \tag{20}$$

When f is a maximal vector for V, (20) is the optimal weighted sensitivity, and $\left|\frac{T\hat{f}}{\hat{f}}\right|^2 = \lambda^2_{max}$. We can then compute the optimal feedback compensator for the case $\beta < 1$ as in the next section.

Remark: As $\Delta \to 0$, $\lambda^2_{max} \to 1$, and as $\Delta \to \infty$, $\lambda^2_{max} \to \frac{1}{\beta}$. This can be seen as follows: From Figure 2, as $\Delta \to 0$, ω_0^2 increases. For large ω_0, $\frac{(1+\beta)\omega_0}{\omega_0^2 - \beta} \to \frac{1+\beta}{\omega_0}$. As $\Delta \to 0$, $\tan(\Delta\omega) \approx \Delta\omega$ over an increasingly larger interval. Thus the smallest solution of (15) is given with increasing accuracy by $\Delta\omega \approx \frac{1+\beta}{\omega}$, or $\omega^2 \approx \frac{1+\beta}{\Delta}$. Since $\lambda^2 = \frac{\omega^2+1}{\omega^2+\beta}$, we find $\lambda^2 \approx \frac{1+\beta+\Delta}{1+\beta+\beta^2\Delta} \approx 1$. When $\Delta \to \infty$, we see from Figure 2 that $\omega_0^2 \to \frac{1}{\beta}$. This gives $\lambda^2 \approx \frac{1+\beta}{\beta+\beta^2} = \frac{1}{\beta}$.

Calculation of Optimal Feedback Compensator for $\beta < 1$.

The optimal weighted sensitivity in (20) can be written

$$\hat{X} = \frac{\omega_0[-s + e^{-s\Delta}(s \cdot \cos\omega_0\Delta - \omega_0\sin\omega_0\Delta)] + e^{-s\Delta}(s \cdot \sin\omega_0\Delta + \omega_0\cos\omega_0\Delta) - \omega_0}{\omega_0[-s + e^{-s\Delta}(s \cdot \cos\omega_0\Delta - \omega_0\sin\omega_0\Delta)] + \beta[-\omega_0 + e^{-s\Delta}(s \cdot \sin\omega_0\Delta + \omega_0\cos\omega_0\Delta)]}$$

$$= \frac{s \cdot (-\omega_0 + e^{-s\Delta}\omega_0\cos\omega_0\Delta + e^{-s\Delta}\sin\omega_0\Delta) + \omega_0 e^{-s\Delta}(\cos\omega_0\Delta - \omega_0\sin\omega_0\Delta) - \omega_0}{s \cdot (-\omega_0 + e^{-s\Delta}\omega_0\cos\omega_0\Delta + e^{-s\Delta}\beta\sin\omega_0\Delta) + \omega_0 e^{-s\Delta}(\beta\cos\omega_0\Delta - \omega_0\sin\omega_0\Delta) - \beta\omega_0}$$

where ω_0 is the smallest solution of (15). We want to find the feedback F which results in this weighted sensitivity. We have $X = W(1-PQ)$ and $F = \frac{Q}{1-PQ}$, so $F = \frac{W-X}{PX}$.

Taking $\hat{X} = \frac{\hat{N}}{\hat{D}}$, we get $\hat{F} = \frac{W\hat{D}-\hat{N}}{P\hat{N}} = \frac{(s+1)\hat{D}-(s+\beta)\hat{N}}{(s+\beta)e^{-s\Delta}P_0\hat{N}}$, with

$\hat{N} = s(k_1 e^{-s\Delta} - \omega_0) + k_2 e^{-s\Delta} - \omega_0$ and $\hat{D} = s(k_3 e^{-s\Delta} - \omega_0) + k_4 e^{-s\Delta} - \beta\omega_0$

$$\hat{F} = \frac{(s+1)\left[s(k_3 e^{-s\Delta} - \omega_0) + k_4 e^{-s\Delta} - \beta\omega_0\right] - (s+\beta)\left[s(k_1 e^{-s\Delta} - \omega_0) + k_2 e^{-s\Delta} - \omega_0\right]}{P_0 e^{-s\Delta}(s+\beta)\left[s(k_1 e^{-s\Delta} - \omega_0) + k_2 e^{-s\Delta} - \omega_0\right]}$$

$$= \frac{-s^2(k_1-k_3)e^{-s\Delta} + s\left[k_3 e^{-s\Delta} + k_4 e^{-s\Delta} - k_2 e^{-s\Delta} - \beta k_1 e^{-s\Delta}\right] + \left[k_4 e^{-s\Delta} - \beta k_2 e^{-s\Delta}\right]}{P_0 e^{-s\Delta} \cdot \left[-s^2(\omega_0 - k_1 e^{-s\Delta}) + s(\beta k_1 e^{-s\Delta} - \beta\omega_0 + k_2 e^{-s\Delta} - \omega_0) + \beta(k_2 e^{-s\Delta} - \omega_0)\right]}$$

with

$$k_1 = \omega_0\cos\omega_0\Delta + \sin\omega_0\Delta, \quad k_2 = \omega_0\cos\omega_0\Delta - \omega_0^2\sin\omega_0\Delta$$

$$k_3 = \omega_0\cos\omega_0\Delta + \beta\sin\omega_0\Delta \quad \text{and} \quad k_4 = \omega_0\beta\cos\omega_0 - \omega_0^2\sin\omega_0.$$

Substituting for the k_is we get

$$\hat{F} = \frac{1}{P_0 e^{-s\Delta}} \cdot$$

$$\frac{(1-\beta)\sin\omega_0\Delta(\omega_0^2+s^2)}{-s^2[\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta-\omega_0e^{s\Delta}]+s[(\omega_0^2-\beta)\sin\omega_0\Delta-(\beta+1)\omega_0\cos\omega_0\Delta+(\beta+1)\omega_0e^{s\Delta}]+\beta\omega_0[\omega_0\sin\omega_0\Delta-\cos\omega_0\Delta+e^{s\Delta}]}$$

$$= \frac{1}{P_0e^{-s\Delta}} \cdot$$

$$\frac{(1-\beta)\sin\omega_0\Delta(\omega_0^2+s^2)}{-s^2(\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta)+s[(\omega_0^2-\beta)\sin\omega_0\Delta-(\beta+1)\omega_0\cos\omega_0\Delta]+\beta\omega_0(\omega_0\sin\omega_0\Delta-\cos\omega_0\Delta)+e^{s\Delta}\omega_0[s^2+(\beta+1)s+\beta]}$$

$$= \frac{1}{P_0e^{-s\Delta}(\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta)} \cdot$$

$$\frac{(1-\beta)\sin\omega_0\Delta(\omega_0^2+s^2)}{-s^2 + \left[\dfrac{\omega_0^2\sin\omega_0\Delta-\omega_0\cos\omega_0\Delta}{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta} - \beta\cdot\dfrac{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta}{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta}\right]s + \beta\cdot\dfrac{\omega_0^2\sin\omega_0\Delta-\omega_0\cos\omega_0\Delta}{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta} + \omega_0e^{s\Delta}\dfrac{s^2+(\beta+1)s+\beta}{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta}}$$

Using (16) we get

$$\hat{F} = \frac{(1-\beta)\sin\omega_0\Delta(\omega_0^2+s^2)}{P_0\left[(\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta)(-s^2+\beta^2)e^{-s\Delta} + \omega_0[s^2+(\beta+1)s+\beta]\right]}$$

$$= \frac{1-\beta}{\omega_0}\sin\omega_0\Delta\cdot P_0^{-1} \cdot$$

$$\frac{s^2+\omega_0^2}{s^2+(\beta+1)s+\beta} \cdot \frac{1}{1+e^{-s\Delta} \cdot \dfrac{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta}{\omega_0} \cdot \dfrac{\beta^2-s^2}{s^2+(\beta+1)s+\beta}}$$

Taking $\zeta_1 = \dfrac{1-\beta}{\omega_0} \cdot \sin\omega_0\Delta$ and $\zeta_2 = \dfrac{\sin\omega_0\Delta+\omega_0\cos\omega_0\Delta}{\omega_0}$ (the expression for

ζ_2 is simplified below) this is

$$\hat{F} = \zeta_1 \cdot P_0^{-1} \cdot \frac{s^2+\omega_0^2}{s^2+(\beta+1)s+\beta} \cdot \frac{1}{1 + e^{-s\Delta} \cdot \zeta_2 \cdot \dfrac{\beta^2-s^2}{s^2+(\beta+1)s+\beta}}.$$
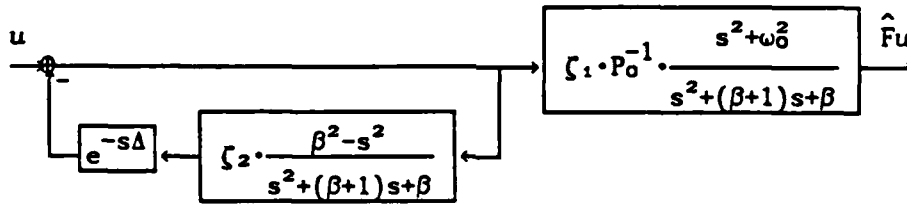
which can be realized as shown in Figure 3.

Figure 3. Realization of Optimal Compensator

Since the optimal PQ is independent of the outer part of P, Q and
thus F will in general be improper. It is necessary to find a proper Q
for the compensator to be physically implementable.

Proper Approximation of Optimal Feedback Compensator.

The only procedures in the literature for this purpose seem to be
given in [Zames and Francis 1983, p. 591] and [Vidyasagar 1985, p. 178].
However the procedure in [Zames and Francis 1983] requires the

evaluation of the term $B_z(\infty)$, where $B_z(s)$ is the Blaschke product formed from plant zeros. In our case there is no Blaschke product involved, but rather a singular inner function. If we interpret $B_z(s)$ to be this inner function, $B_z(\infty)$ is not defined. In fact this procedure does not work for our case.

The procedure in [Vidyasagar 1985] does not work for our case either. The essence of the difficulty is the same as in the Zames-Francis procedure — the inner factor of the plant is not continuous at infinity.

An explanation of why these techniques do not work is given in [Flamm 1985b]. The analysis there does suggest a means to accomplish the approximation. A full discussion will appear in Part III of this report, but the graphical argument presented in [Flamm 1985b] to show why the other procedures do not work motivates the following approach.

The essential idea is to roll-off the ideal Q-parameter with a transfer function for which the Bode magnitude plot has slope less than 1, so as to limit the phase deviation due to the roll-off, until sufficient attenuation has been obtained. This this can be accomplished, for example, with a lead-lag network which approximates such compensation by having average slope magnitude less than 1.

We note that any stable Q-parameter results in a stable closed loop system, so that this roll-off technique preserves stability just as the procedures for the isolated right half plane zero case do.

Stability of Optimal Feedback Compensator.

The compensator is of the form

$$\hat{F} = \zeta_1 \cdot P_0^{-1} \cdot \frac{s^2 + \omega_0^2}{(s+\beta)} \cdot \frac{-1}{(s+1) + e^{-s\Delta}\zeta_2(s-\beta)}.$$

Since P_0 is minimum phase by assumption, the problem amounts to determining the stability of the term $\dfrac{-1}{(s+1) + e^{-s\Delta}\zeta_2(s-\beta)}$. Equivalently, we ask whether $1 + e^{-s\Delta}\zeta_2 \dfrac{s-\beta}{s+1}$ has zeros in the right half plane. We answer this question by proving that $1 + e^{-s\Delta}\zeta_2 \dfrac{s-\beta}{s+1}$ has finitely many zeros in the closed left half plane, and then conclude by appealing to Picard's theorem that $1 + e^{-s\Delta}\zeta_2 \dfrac{s-\beta}{s+1}$ has infinitely many zeros in the right half plane.

First we note that $\left| e^{-s\Delta} \right| > 1$ for s in the left half plane, and $\left| e^{-s\Delta} \right| < 1$ in the right half plane. Now all zeros must satisfy $e^{-\Re e(s\Delta)} \cdot \left| \dfrac{s-\beta}{s+1} \right| = \dfrac{1}{|\zeta_2|}$, and therefore all closed left half plane zeros satisfy $\left| \dfrac{s-\beta}{s+1} \right| \leq \dfrac{1}{|\zeta_2|}$. So all closed left half plane zeros lie on or inside the intersection of the ellipse $\left| \dfrac{s-\beta}{s+1} \right| = \dfrac{1}{|\zeta_2|}$ with the closed left half plane. See Figure 4.
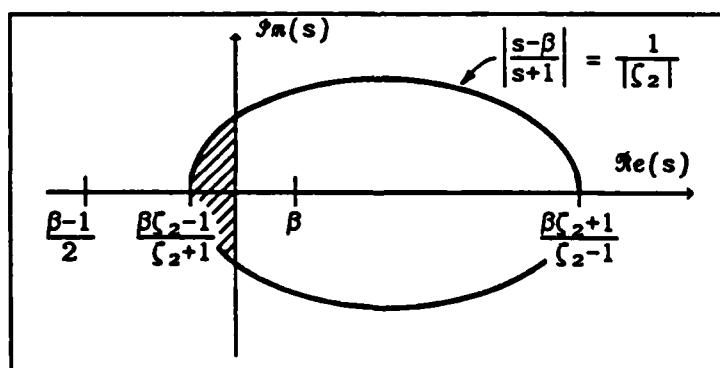

Figure 4. Region for possible left half plane zeros.

Thus all closed left half plane zeros lie in a compact region, and we conclude from analyticity that there are only finitely many in that region. Picard's theorem tells us that there are infinitely many zeros, so that we must conclude that there are infinitely many in the open right half plane.

We can actually get more detailed information about the distribution of the right half plane zeros without much more trouble. As stated above, the right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| > \frac{1}{|\zeta_2|}$. As $|s| \to \infty$, $\left|\frac{s-\beta}{s+1}\right| \to 1$. Therefore as $|s| \to \infty$, the zero set $\{z_t\}$ approaches the line $|e^{-s\Delta}| = \frac{1}{|\zeta_2|}$, which is the same as $\Re e(s) = \frac{\ln|\zeta_2|}{\Delta}$, and $\Im m(z_t) \to (2n+1)\pi$. Also, since right half plane zeros must satisfy $\left|\frac{s-\beta}{s+1}\right| < 1$ (by comparing distances from β and from the point -1), we have $|\zeta_2| \cdot \left|\frac{s-\beta}{s+1}\right| < |\zeta_2|$. Then since $|e^{-s\Delta}| \cdot \left|\frac{s-\beta}{s+1}\right| = \frac{1}{|\zeta_2|}$, $|e^{s\Delta}| < |\zeta_2|$, and we conclude $\Re e(s) < \frac{\ln|\zeta_2|}{\Delta}$ for these zeros.

The fact that $H(s) = (s+1)+e^{-s\Delta}\zeta_2(s-\beta)$ has right half plane zeros can also be obtained from results on the distribution of zeros of entire functions. (See [Levin 1980] Chapter 7, §4, p. 323, Example 1.) For our purposes it is more convenient to refer to the earlier work [Pontryagin 1955].

According to theorem 6 in this latter paper it is sufficient to show that the function $G(y) = y \cdot \cos(y\Delta) + \sin(y\Delta) + \zeta_2 \cdot y$ has zeros which

are not purely real.[3] To see this all we need is to recognize that G
has infinitely many zeros. (To show this we can use theorem 3 in the
same paper.) Setting $G(y) = 0$ we get

$$-y \cdot \cos(y\Delta) = \sin(y\Delta) + \zeta_2 \cdot y.$$

For real values of y, the left hand side of this equation has the lines
$z = \pm y$ for an envelope, whereas the right hand side oscillates with
deviation 1 about the line $z = \zeta_2 \cdot y$. Since $|\zeta_2| > 1$,[4] for some value y_o
there are no more real zeros for $|y| > y_o$.

Acknowledgement

The work described in this report is an elaboration of ideas pre-
sented in [Flamm 1985a]. We have generalized this work in the direc-
tions indicated on pp. 6-7 above, and we are preparing additional
reports containing the details. Subsequent to the drafting of this
report we have learned from George Zames that similar work has been done
independently by Foias, Tannenbaum and Zames. At the time of this writ-
ing we have not had the opportunity to examine their work.

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[3]Theorem 6 says that if we evaluate H(s) on the imaginary axis, and
split the resulting $H(iy)$ ($y \in \mathbb{R}$) into real and imaginary parts,
$H(iy) = F(y) + iG(y)$ with $F(y)$ and $G(y)$ taking only real values,
then necessary and sufficient conditions for H to have all its zeros
in the left half plane are that (i) F and G have only real zeros,
that (ii) these zeros alternate, and (iii) for at least one value of
y, $G'(y)F(y) - F'(y)G(y) > 0$. Thus we need only show that G has a
complex zero to establish that H has a right half plane zero.

[4]From (15), $\dfrac{\sin \omega_0 \Delta}{\omega_0} = \dfrac{1+\beta}{\omega_0^2 - \beta} \cos \omega_0 \Delta$. Using this in the defini-tion of ζ_2,

$\zeta_2 = \dfrac{1+\omega_0^2}{\omega_0^2 - \beta} \cos \omega_0 \Delta$. Therefore $\zeta_2^2 = \left[\dfrac{1+\omega_0^2}{\omega_0^2 - \beta}\right]^2 \cdot \dfrac{1}{1 + \tan^2 \omega_0^2 \Delta} = \dfrac{\omega_0^2 + 1}{\omega_0^2 + \beta}$, where we

have used (15) to substitute for $\tan(\omega_0 \Delta)$, and simplified.

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